Integration of this equation throughout an arbitrary volume v gives

$$\int_v \sigma E^2\,dv = -\int_v \left(\frac{\epsilon}{2}\frac{\partial E^2}{\partial t} + \frac{\mu}{2}\frac{\partial H^2}{\partial t}\right) dv - \oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}$$

where the last term has been converted to an integral over the surface of v by use of the divergence theorem.

The integral on the left has the units of watts and is the usual ohmic term representing energy dissipated per unit time in heat. This dissipated energy has its source in the integrals on the right. Because $\epsilon E^2/2$ and $\mu H^2/2$ are the densities of energy stored in the electric and magnetic fields, respectively, the volume integral (including the minus sign) gives the decrease in this stored energy. Consequently, the surface integral (including the minus sign) must be the rate of energy entering the volume from outside. A change of sign then produces the *instantaneous rate of energy leaving the volume*:

$$P(t) = \oint_S (\mathbf{E}\times\mathbf{H})\cdot s\mathbf{S} = \oint_S \mathcal{P}\cdot d\mathbf{S}$$

where $\mathcal{P} = \mathbf{E}\times\mathbf{H}$ is the *Poynting vector*, the instantaneous rate of energy flow per unit area at a point.

In the cross product that defines the Poynting vector, the fields are supposed to be in real form. If, instead, $\mathbf{E}$ and $\mathbf{H}$ are expressed in complex form and have the common time-dependence $e^{j\omega t}$, then the time-average of $\mathcal{P}$ is given by

$$\mathcal{P}_{\text{avg}} = \tfrac{1}{2}\,\text{Re}\,(\mathbf{E}\times\mathbf{H}^*)$$

where $\mathbf{H}^*$ is the complex conjugate of $\mathbf{H}$. This follows the *complex power* of circuit analysis, $\mathbf{S} = \frac{1}{2}\mathbf{VI}^*$, of which the power is the real part, $P = \frac{1}{2}\,\text{Re}\,\mathbf{VI}^*$.

For plane waves, the direction of energy flow is the direction of propagation. Thus the Poynting vector offers a useful, coordinate-free way of specifying the direction of propagation, or of determining the directions of the fields if the direction of propagation is known. This can be particularly valuable where incident, transmitted, and reflected waves are being examined.

Solved Problems

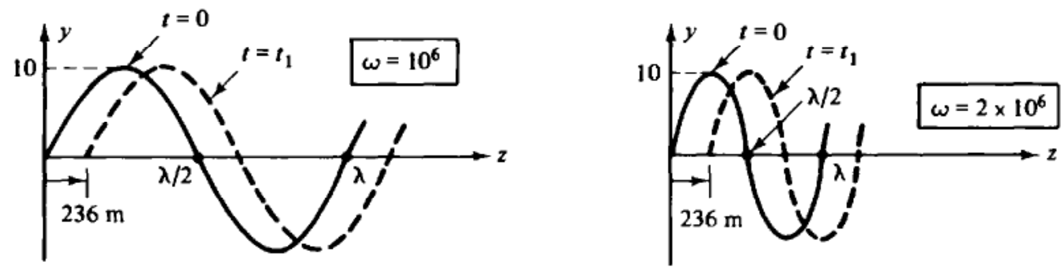
14.1. A traveling wave is described by $y = 10\sin(\beta z - \omega t)$. Sketch the wave at $t = 0$ and at $t = t_1$, when it has advanced $\lambda/8$, if the velocity is 3×10^8 m/s and the angular frequency $\omega = 10^6$ rad/s. Repeat for $\omega = 2\times 10^6$ rad/s and the same t_1.

The wave advances λ in one period, $T = 2\pi/\omega$. Hence

$$t_1 = \frac{T}{8} = \frac{\pi}{4\omega}$$

$$\frac{\lambda}{8} = ct_1 = (3\times 10^8)\frac{\pi}{4(10^6)} = 236\ \text{m}$$

The wave is shown at $t = 0$ and $t = t_1$ in Fig. 14-9(a). At twice the frequency, the wavelength λ is one-half, and the phase shift constant β is twice, the former value. See Fig. 14-9(b). At t_1 the wave has also advanced 236 m, but this distance is now $\lambda/4$.



(a) (b)

Fig. 14-9

14.2. In free space, $\mathbf{E}(z, t) = 10^3 \sin(\omega t - \beta z)\mathbf{a}_y$ (V/m). Obtain $\mathbf{H}(z, t)$.

Examination of the phase, $\omega t - \beta z$, shows that the direction of propagation is $+z$. Since $\mathbf{E} \times \mathbf{H}$ must also be in the $+z$ direction, $\mathbf{H}$ must have the direction $-\mathbf{a}_x$. Consequently,

$$\frac{E_y}{-H_x} = \eta_0 = 120\pi\,\Omega \qquad \text{or} \qquad H_x = -\frac{10^3}{120\pi}\sin(\omega t - \beta z) \quad \text{(A/m)}$$

and

$$\mathbf{H}(z, t) = -\frac{10^3}{120\pi}\sin(\omega t - \beta z)\mathbf{a}_x \quad \text{(A/m)}$$

14.3. For the wave of Problem 14.2 determine the propagation constant γ, given that the frequency is $f = 95.5$ MHz.

In general, $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$. In free space, $\sigma = 0$, so that

$$\gamma = j\omega\sqrt{\mu_0\epsilon_0} = j\left(\frac{2\pi f}{c}\right) = j\frac{2\pi(95.5 \times 10^6)}{3 \times 10^8} = j(2.0)\ \text{m}^{-1}$$

Note that this result shows that the attenuation factor is $\alpha = 0$ and the phase-shift constant is $\beta = 2.0$ rad/m.

14.4. Examine the field

$$\mathbf{E}(z, t) = 10\sin(\omega t + \beta z)\mathbf{a}_x + 10\cos(\omega t + \beta z)\mathbf{a}_y$$

in the $z = 0$ plane, for $\omega t = 0, \pi/4, \pi/2, 3\pi/4$ and π.

The computations are presented in Table 14-1.

Table 14-1

ωt	$E_x = 10\sin\omega t$	$E_y = \cos\omega t$	$\mathbf{E} = E_x\mathbf{a}_x + E_y\mathbf{a}_y$
0	0	10	$10\mathbf{a}_y$
$\frac{\pi}{4}$	$\frac{10}{\sqrt{2}}$	$\frac{10}{\sqrt{2}}$	$10\left(\frac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}\right)$
$\frac{\pi}{2}$	10	0	$10\mathbf{a}_x$
$\frac{3\pi}{4}$	$\frac{10}{\sqrt{2}}$	$\frac{-10}{\sqrt{2}}$	$10\left(\frac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}\right)$
π	0	-10	$10(-\mathbf{a}_y)$